

Ordinary differential equations VS difference equations

Sandra Pinelas
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Difference equations vs. differential equations

The **seasonal breeders** are animal species that successfully mate only during certain times of the year.

The model is given by a **difference equation**

$$N_{t+1} = \lambda N_t \quad \text{or} \quad N_{t+n} = \lambda^n N_t$$

An example:

Imagine a population that initially has $N_0 = 10$ individuals and a growth rate of $\lambda = 1.1$ per week.

After 52 weeks (approximately 1 year) the equation

$$N_{52} = \lambda^{52} N_0$$

shows that the population is 1420 and, after 2 years, the population will be 201.762 individuals.

Polar bear: April/May



Difference equations vs. differential equations

The **continuous breeders**, like humans, that mate year-round.

The **differential equation** will then be rewritten as

$$N'(t) = r N(t).$$



$r = b - d$ is the
Malthusian parameter

The solution is

$$N(t) = N(t_0) e^{r(t-t_0)},$$

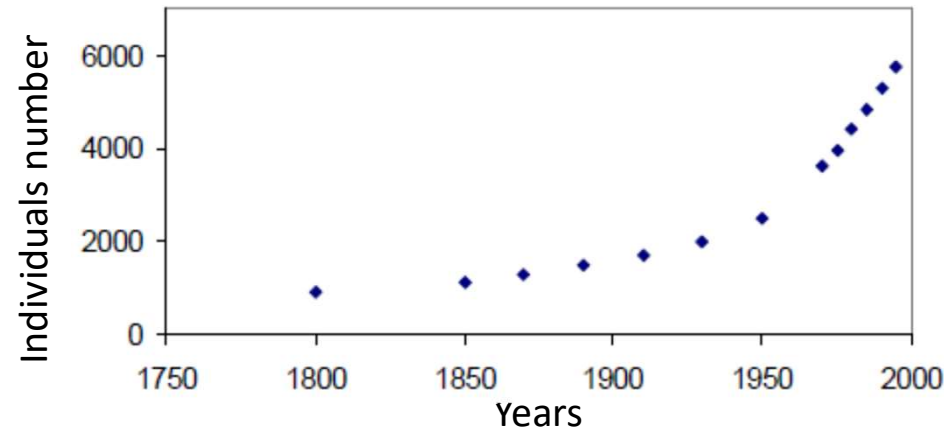
$N(t_0)$ is the greatness of the population at the beginning of the time.

Human being



Difference equations vs. differential equations

An example:



The population in 1950 was 2520 millions and in 1995 was 5720 millions, so

$$5720 = 2520e^{45r} \quad \Leftrightarrow \quad r = \frac{1}{45} \ln\left(\frac{5720}{2520}\right) = 0,018$$

⇒ annual increment rate: $\lambda = e^{0,018} = 1,01859 = 1,859\%$

How long does the population take to double with such a rate? $n = \frac{\ln 2}{\ln 1,01859} = 37,6$ years! (in 2033)

The logistic equation

The logistic differential equation is given by:

$$N'(t) = r N(t) \left(1 - \frac{N(t)}{K} \right)$$

K is the
"carrying capacity"



$$\Leftrightarrow \ln(N(t)) - \ln \left(1 - \frac{N(t)}{K} \right) - \left(\ln(N(0)) - \ln \left(1 - \frac{N(0)}{K} \right) \right) = rt$$

$$\Leftrightarrow \frac{N(t)}{1 - \frac{N(t)}{K}} = \frac{N_0}{1 - \frac{N_0}{K}} e^{rt} \quad \Leftrightarrow \quad N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}.$$

Equilibrium point

We can get information about the status of the dynamical system finding the equilibrium point.

In fact for a differential equation given by

$$x'(t) = f(x(t)), \quad t > t_0$$

We obtain the equilibrium when $x'(t) = 0$.

So, for example to the logistic equation we get

$$\begin{aligned} N'(t) = 0 &\Leftrightarrow rN(t) \left(1 - \frac{N(t)}{K}\right) = 0 \\ &\Leftrightarrow N(t) = 0 \quad \vee \quad N(t) = K \end{aligned}$$

$$\lim_{t \rightarrow +\infty} N(t) = \lim_{t \rightarrow +\infty} \frac{KN(t_0)}{N(t_0) + (K - N(t_0))e^{-rt}} = K.$$

Equilibrium point

An example:

$$N'(t) = 0,5N(t) \left(1 - \frac{N(t)}{20} \right)$$

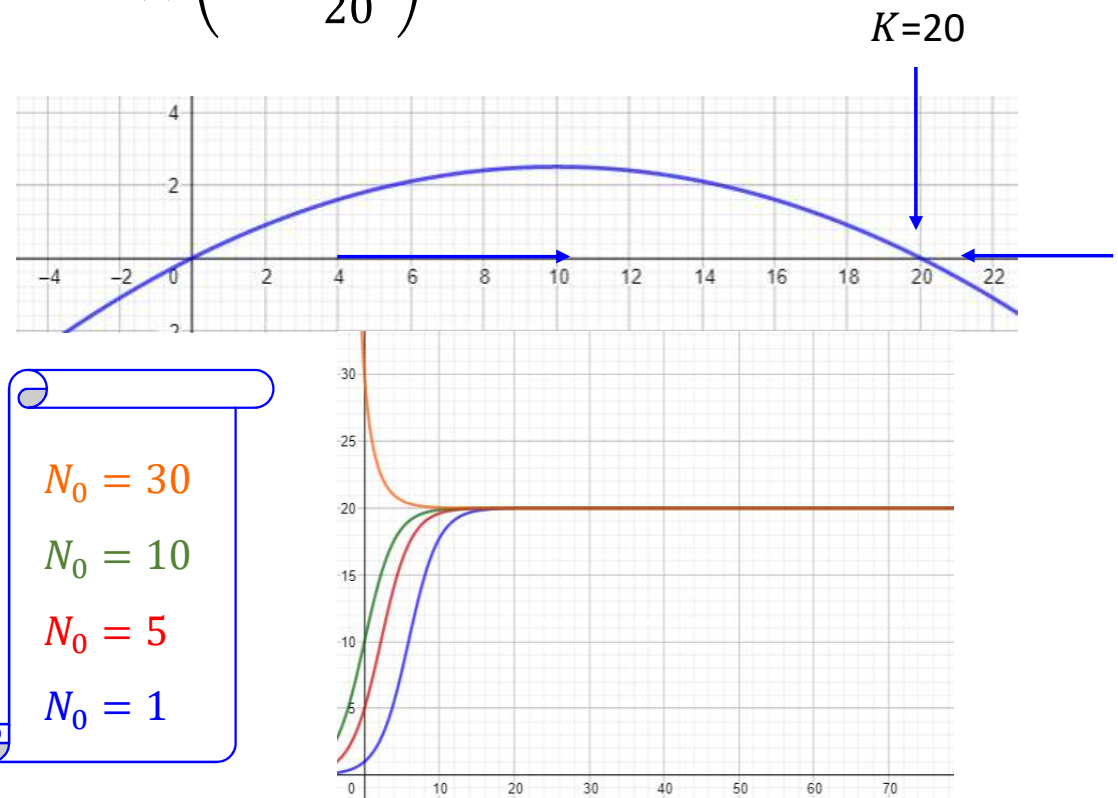
So, the equilibrium points are

$$N(t) = 0 \quad \vee \quad N(t) = 20$$

and

$$N(t) = \frac{20N(t_0)}{N(t_0) + (20 - N(t_0))e^{-r}}.$$

$$N_0 = 30$$
$$N_0 = 10$$
$$N_0 = 5$$
$$N_0 = 1$$



Allee effect

For certain populations, it is permissible to have a minimum density below which the likelihood of effective encounters between individuals (or reproductive cells) of both sexes is so low that the population cannot return their density to the same value.

This phenomenon is represented by the differential equation.

$$N'(t) = r (N(t) - E) \left(1 - \frac{N(t)}{K}\right)$$

where E represents the critical minimum density.



Regulated growth models: the Allee effect

If the population grows slightly above E , the likelihood of effective encounters between individuals of both sexes increases and, consequently, $N'(t) > 0$. The population then begins to grow to K .

If the population drops to a density slightly below E , the likelihood of encounters decreases sufficiently that $N'(t) < 0$, and the population irreversibly tends to extinction.

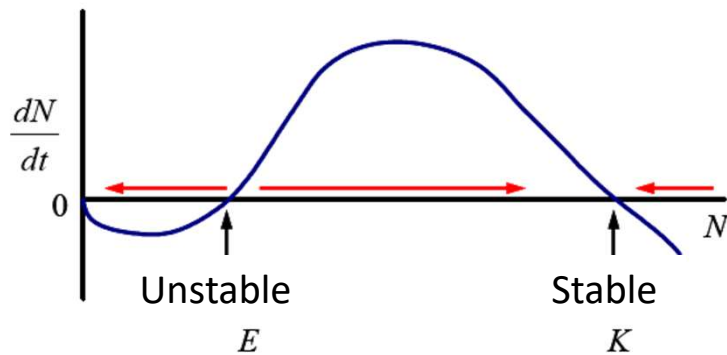
The solution is

$$N(t) = E + \frac{(K - E)(N_0 - E)}{(N_0 - E) + (K - N_0)e^{-rt}}$$

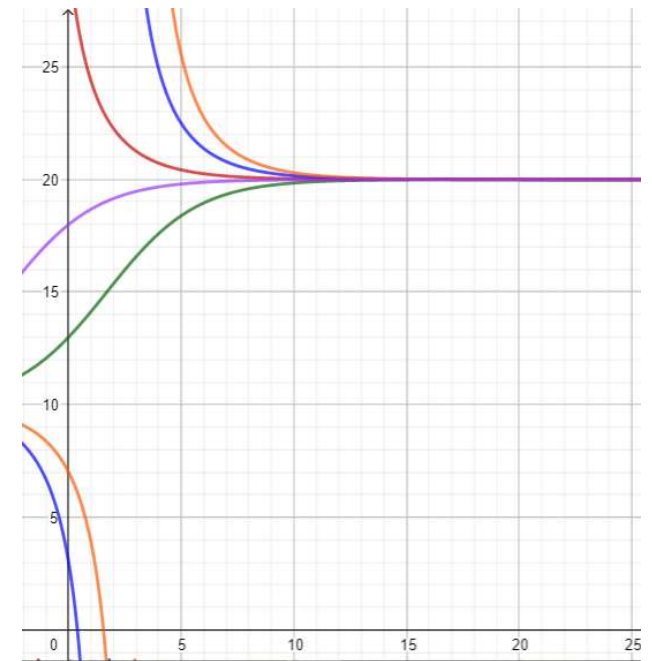
Regulated growth models: the Allee effect

An example:

$$N'(t) = r (N(t) - E) \left(1 - \frac{N(t)}{K} \right) \quad \Leftrightarrow \quad N(t) = E + \frac{(K - E)(N_0 - E)}{(N_0 - E) + (K - N_0)e^{-rt}}$$



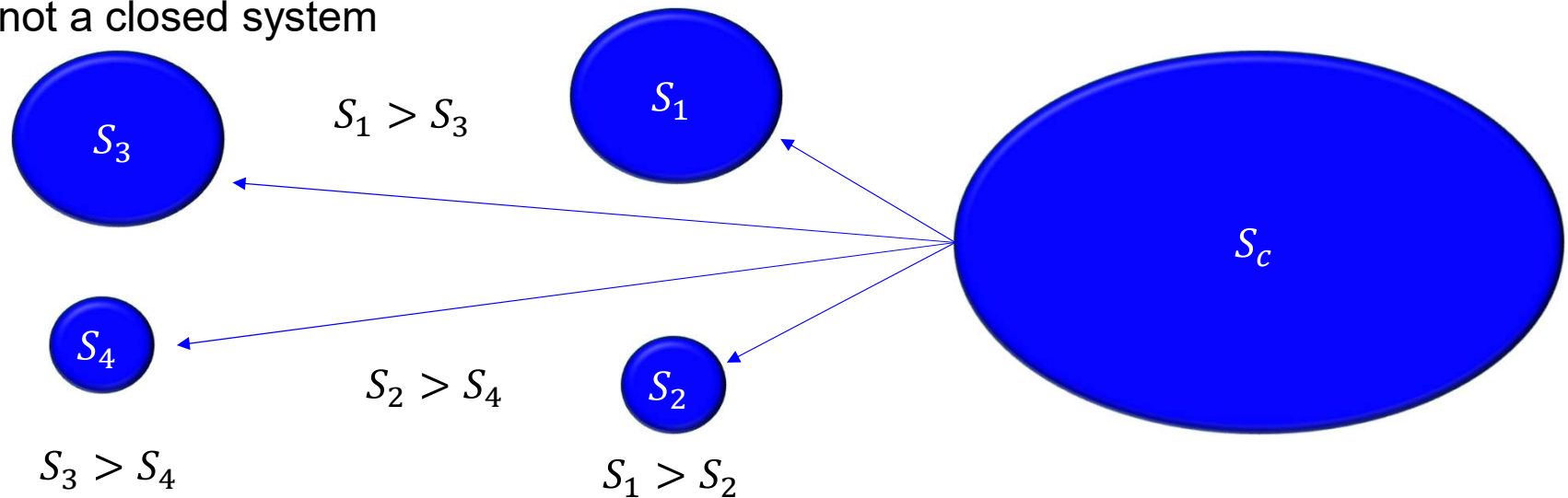
$$\begin{aligned} E &= 10 \\ K &= 20 \\ N_0 &= 3 \\ N_0 &= 7 \\ N_0 &= 13 \\ N_0 &= 18 \\ N_0 &= 30 \end{aligned}$$



Metapopulations

This theory was based on several assumptions:

- Island communities are poorer in species than equivalent continental communities ($S_c > S_1$);
- This wealth increases with the size of the island;
- This wealth decrease with the increasing isolation of the island.
- Islands are not a closed system



Metapopulations: some examples

The **Glanville fritillary** (**Melitaea cinxia**) is an endangered butterfly species that disappeared from Finland in the late 1970s and now occurs in Finland only on the islands of Åland and a few surrounding Islands.



One of the best studied metapopulations are the host plants Plantago and Veronica that occur in dry fields of Åland and it are the potential habitat patches for Melitaea cinxia.



Metapopulations: Levins's model

Levins's (1969) model is a simple way of describing the habitat occupation dynamics (dp/dt).

In this model:

- the risk of extinction (e) is considered constant and equal in all spots;
- the extinction rate at time t is then: eP
- the possibility of colonization is considered proportional to the proportion of occupied spots (source of colonizers): cP
- the colonization rate at time t is: $cP(1 - P)$

So
$$P'(t) = cP(1 - P) - eP$$

Metapopulation reaches equilibrium

$$P = 1 - e/c.$$

Metapopulation should be maintained only if $c > e$

A metapopulation tends to the extinction if:

- the average size of the fragments decreases;
- the density of the fragments decreases (increased isolation).

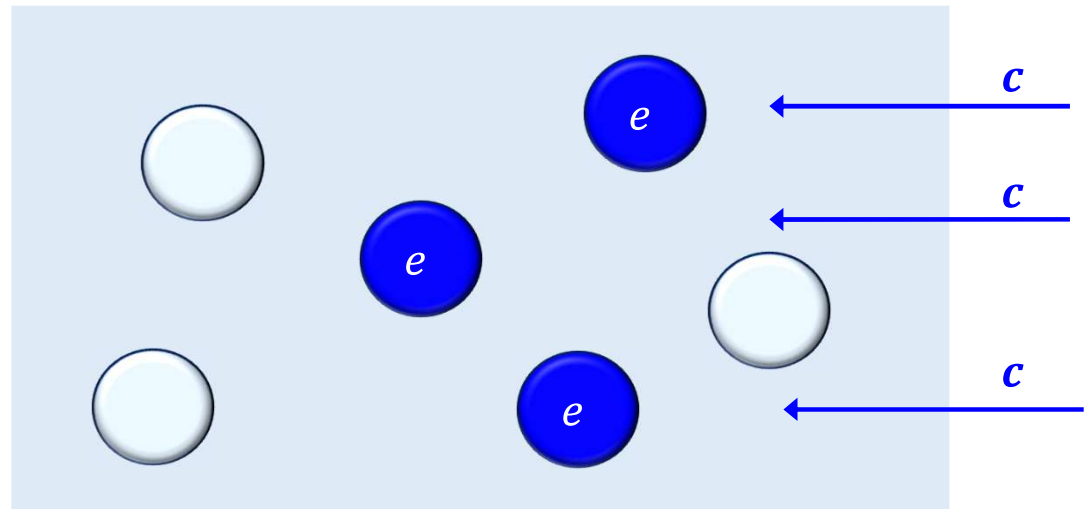
Metapopulations: Boorman & Levitt Model

According with the Boorman & Levitt Model (1973), the colonization depends on an external source of propagules (a “continent”) and extinction is independent of neighboring patches.

$$P'(t) = c(1 - P) - eP$$

Equilibrium point: $P = c/(e + c)$

- if $e = 0$ then $P = 1$;
- if $e > 0$ then $P < 1$;
- if $c = 0$ then $P = 0$;
- if $c = 1$ then $P < 1$.



Metapopulations: Hanski's model

In the Hanski's model (1982) is considered the rescue effect.

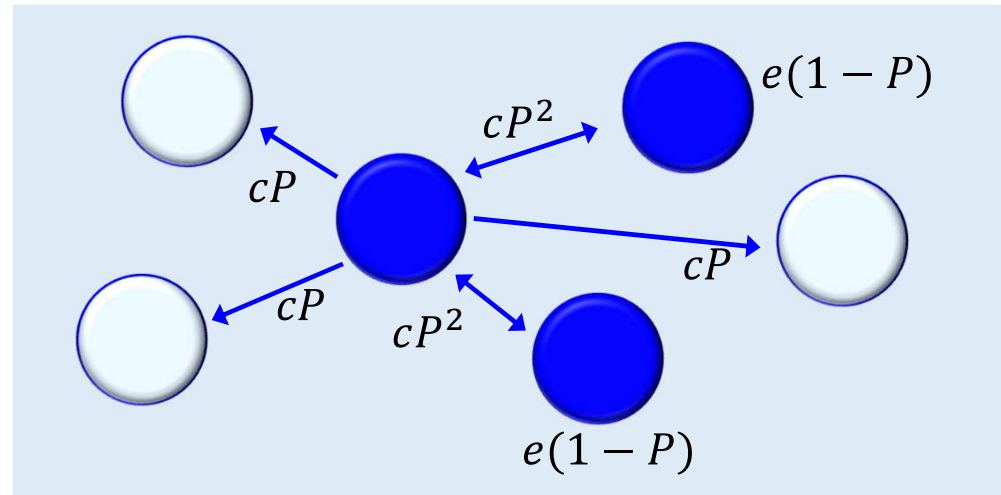
Decreased risk of extinction due to population increase;

Colonization and extinction depend on regional resources (P)

$$P'(t) = cP(1 - P) - eP(1 - P)$$

Equilibrium point: $c = e$ for any P

- if $c > e$ then P tends to 1
- if $c < e$ then P tends to 0



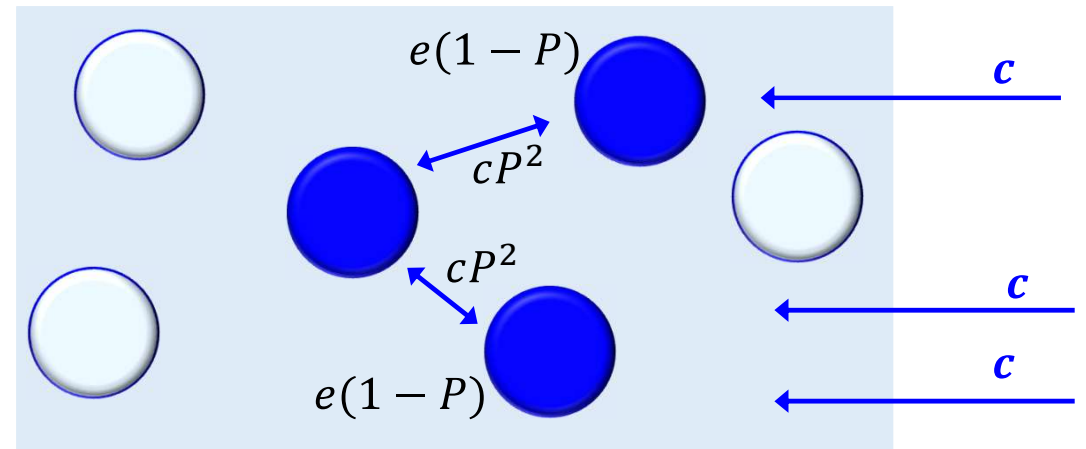
Metapopulations: Gotelli's Model

For the Gotelli's Model (1991), the colonization depends on an external source of propagules (a continent) and the extinction dependent on neighboring patches.

$$P'(t) = c(1 - P) - eP(1 - P)$$

Equilibrium point: $P = c/e$

- if $c > e$ then $P > 1$
- if $c < e$ then $P < 1$
- if $c = 0$ then $P = 0$



(contradictory model, because if there is influence neighboring patches on extinction should also be in colonization)

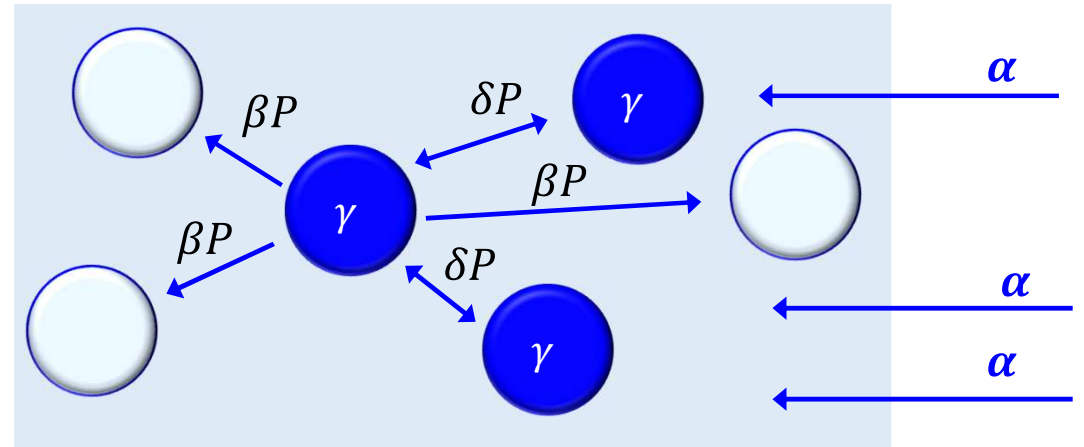
Metapopulations: Gotelli & Kelley

With Gotelli & Kelley (1993) we reach a more realistic model

$$P'(t) = (\alpha + \beta P)(1 - P) - (\gamma - \delta P) P$$

Where:

- α is the contribution of external propagules,
- β is the indicator of the colonization force,
- γ is the intrinsic measure of extinction of a fragment
- δ is the "saving" effect.



Point and equilibrium: $(\beta - \delta)P^2 + (\gamma + \alpha - \beta)P - \alpha = 0$

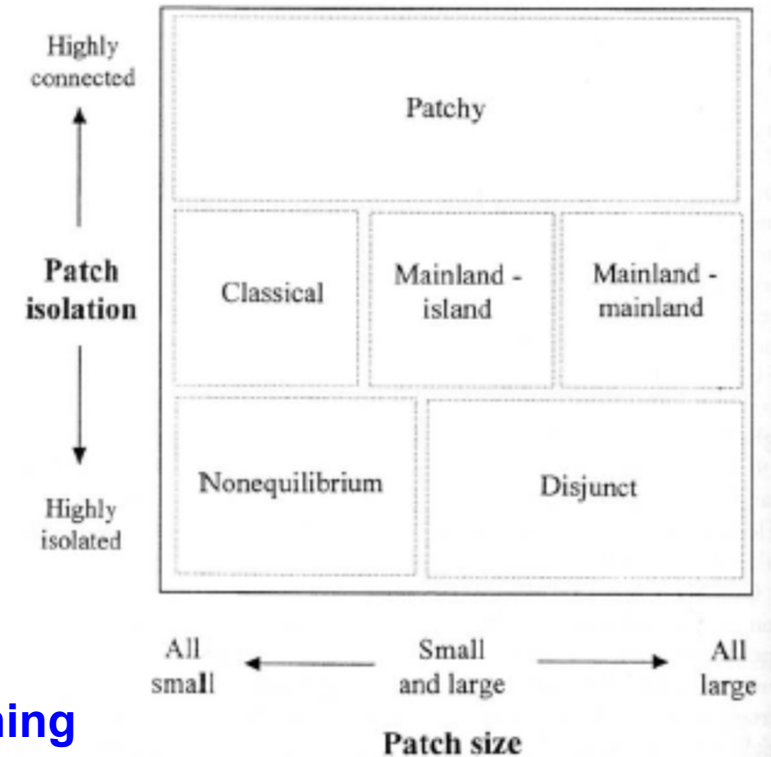
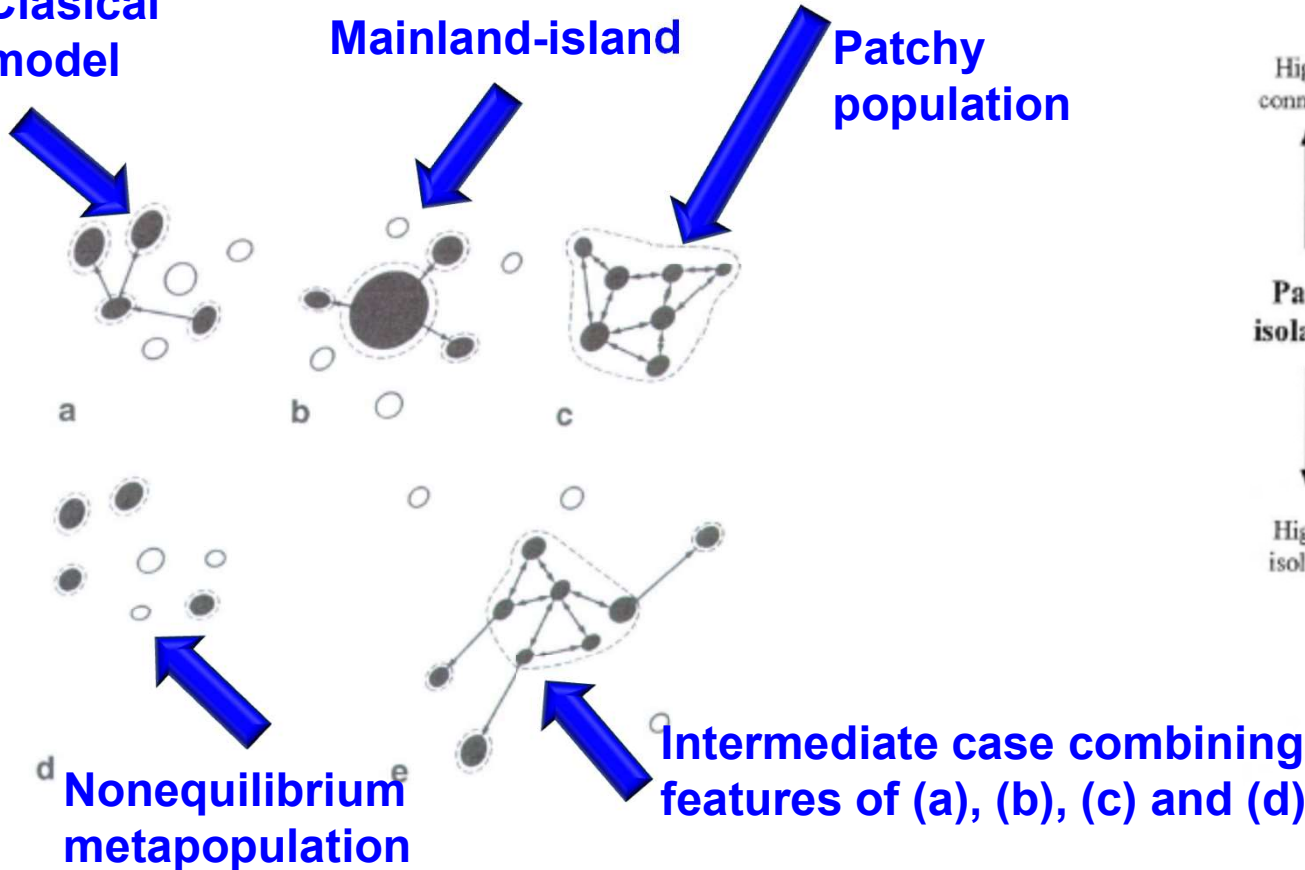
Metapopulations: the cases that should be considerate

In 1997, the concept of metapopulation was expanded to include other types of fragmented populations.

Classical model

Mainland-island

Patchy population



Difference equation: logistic model

The discrete form of the logistic equation is the difference equation

$$N(t + 1) = N(t) + rN(t) \left(1 - \frac{N(t)}{K} \right)$$


$$\Delta N(t + 1) = N(t + 1) - N(t)$$

Difference equation: Ricker's model

The Ricker's (1954) model can be used to predict the number of fish that will be present in a fishery

From the solution of the model of the instantaneous variation

of the number "larvae"

mortality by predation

$$L'(t) = -(d_1 N(t) + d_2)L(t)$$

we reach to the difference equation

Natural mortality

$$N(t + 1) = bN(t)e^{-(d_1 N(t) + d_2)}$$

And the Ricker's model

$$N(t + 1) = N(t)e^{r\left(1 - \frac{N(t)}{K}\right)}$$



Difference equation: Hassel's model

Subsequent work has derived the model under other assumptions such as scramble competition, within-year resource limited competition or even as the outcome of source-sink Malthusian patches linked by density-dependent dispersal.

The Ricker model is a limiting case of the Hassell model which takes the form

$$N(t + 1) = K \frac{N(t)}{(1 + cN(t))^\delta}$$

The delay effect: logistic equation

Population density is unlikely to elicit an instant response to the per capita growth rate.

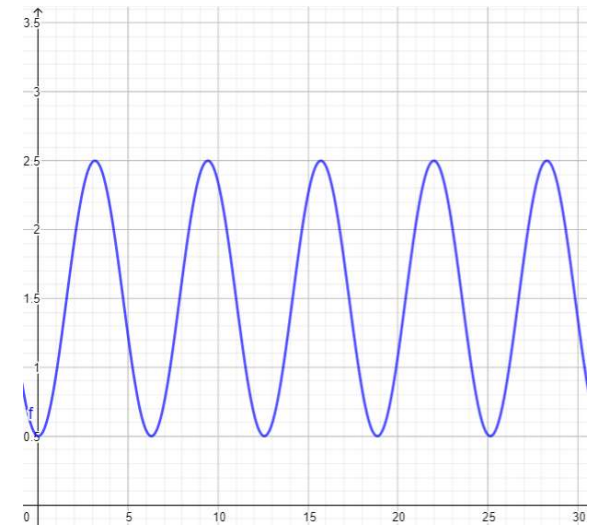
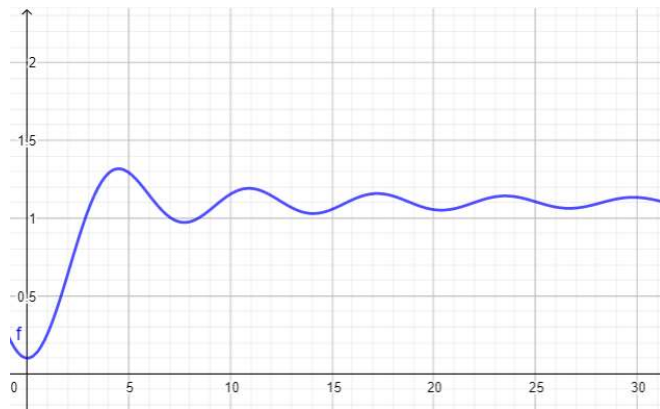
For example, the effect of food scarcity available to young immatures can only be felt later when they reach maturity expressing lower fertility rates.

By designating τ the delay interval we get the delayed logistic equation:

$$N'(t) = rN(t) \left(1 - \frac{N(t - \tau)}{K} \right)$$

or

$$N'(t) = rN(t) \left(\frac{K - N(t - \tau)}{K + crN(t - \tau)} \right)$$



The delay effect: Nicholson's blowflies

The delay differential equation

$$N'(t) = -\delta N(t) + PN(t - \tau)e^{-aN(t - \tau)}.$$

was used by Gurney, S. P. Blythe and R. M. Nishbet (1980)

and it was used to describe the dynamics of Nicholson's blowflies.

Here:

- P is the maximum per capita daily egg production rate
- $1/a$ is the size at which the population reproduces at its maximum rate
- δ is the per capita daily adult death rate
- τ is the generation time



The delay effect: Nicholson's blowflies

The delay differential equation

$$N'(t) = -\delta N(t) + PN(t - \tau)e^{-aN(t - \tau)}$$

has a unique solution for each initial condition

$$N(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

For $P > \delta$ the positive equilibrium point is given by

$$N^* = \frac{1}{a} \ln(P/\delta).$$

The delay effect: survival of red blood cells

The delay differential equation

$$N'(t) = -\mu N(t) + pe^{-\gamma N(t-\tau)}.$$

has been used by Wazewska-Czyzewska and Lasota (1988)

as a model for the survival of red blood cells in an animal.

Here:

- μ is the probability of death of a red blood cell
- p and γ are positive constants and are related to the production of red blood cells per unit of time
- τ is the time required to produce a red blood cell.



The delay effect: survival of red blood cells

The delay differential equation

$$N'(t) = -\mu N(t) + p e^{-\gamma N(t-\tau)}.$$

has a unique solution for each initial condition

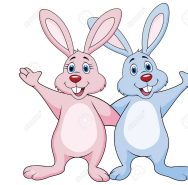
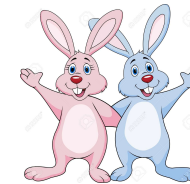
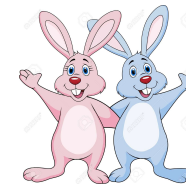
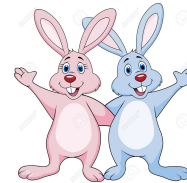
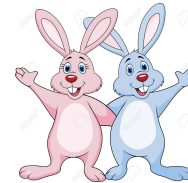
$$N(t) = \varphi(t), \quad -\tau \leq t \leq 0$$

For $P > \delta$ the positive equilibrium point is given by

$$N^* = \frac{p}{\mu} e^{-\gamma N^*}.$$

Fibonacci difference equation

- For $t = 0$ we have a couple of baby rabbits
- For $t = 1$ the babies grow and are ready to have bays in one month
- For $t = 2$ we have a couple of adult rabbits and a couple of baby rabbits
- For $t = 3$ we have the babies grow and the old couple have 2 one more couple
- For $t = 4$ we have the babies grow and the old couples have 2 one more couple



Fibonacci difference equation

The Fibonacci difference equation is given by

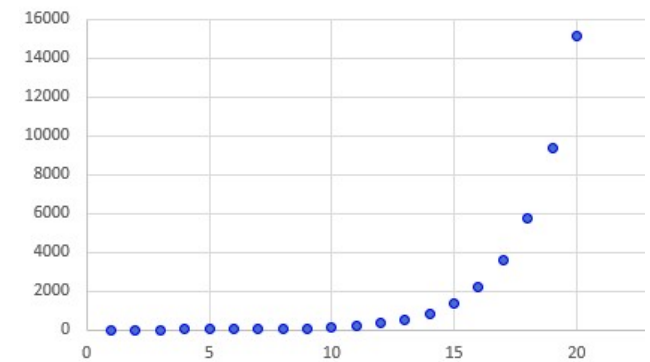
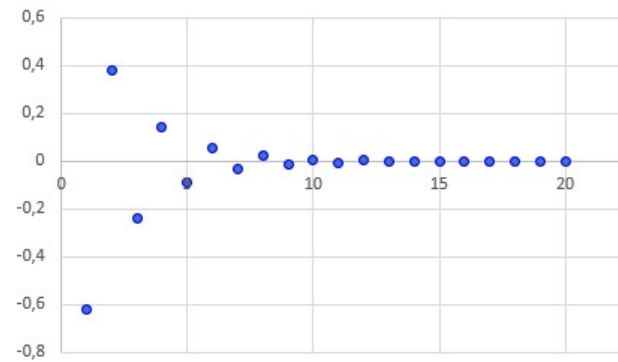
$$N(t) = N(t - 1) + N(t - 2).$$

has an oscillatory solution








$$N(t) = \left(\frac{1 - \sqrt{5}}{2} \right)^t$$

and a nonoscillatory solution








$$N(t) = \left(\frac{1 + \sqrt{5}}{2} \right)^t$$



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